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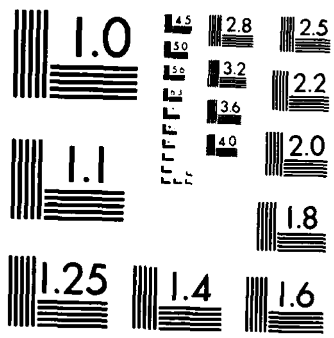

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MRC Technical Summary Report #2946

ON THE METHOD OF TANGENT HYPERBOLAS IN BANACH SPACES

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ABSTRACT

As an application of the technique employed by the author in a series of papers [13] - [16], some results are established concerning convergence of the method of tangent hyperbolas for solving nonlinear equations in Banach spaces as well as existence and uniqueness of solution. The results are compared with those obtained by Döring [4].

AMS (MOS) Subject Classifications: 65G99, 65J15

Key Words: method of tangent hyperbolas, nonlinear equations in Banach spaces, convergence theorem, error estimates.

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SIGNIFICANCE AND EXPLANATION

Finding sharp error bounds for iterative solutions of nonlinear equations is one of the important subjects in numerical analysis. This paper gives a simple technique for finding sharp error bounds for the method of tangent hyperbolas in a Banach space.

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The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON THE METHOD OF TANGENT HYPERBOLAS IN BANACH SPACES

Tetsuro Yamamoto*

1. Introduction

Let X and Y be Banach spaces and $F : D \subseteq X \to Y$ be twice Fréchet differentiable in an open convex domain $D_0 \subseteq D$. Then the method of tangent hyperbolas for solving the equation

$$F(x) = 0 \tag{1.1}$$

is defined as follows:

$$x_{n+1} = x_n - A(x_n)^{-1}F(x_n) \quad , \quad n \geq 0 \quad , \tag{1.2}$$

where $x_0 \in D_0$ and $A(x)$ is defined by

$$A(x) = F'(x) - \frac{1}{2} F''(x)F'(x)^{-1}F(x) \quad , \tag{1.3}$$

provided that $A(x_n)^{-1}$ as well as $F'(x_n)^{-1}$ exists at each step. This is a variant of Newton's method and the procedure may also be written as

$$F'(x_n) + F'(x_n)c_n = 0 \quad , \tag{1.4a}$$

$$F(x_n) + F'(x_n)d_n + \frac{1}{2} F''(x_n)c_n d_n = 0 \quad , \tag{1.4b}$$

$$x_{n+1} = x_n + d_n \quad , \quad n \geq 0 \quad . \tag{1.4c}$$

There is much literature concerning convergence and error estimates for the method. Among others, Mertvecova [9] and Safiev [12] gave convergence theorems of Newton-Kantorovich type for (1.2), whose proofs are based upon recurrence relations similar to Kantorovich's one for Newton's method (cf. [6], [10]). The more detailed and sophisticated discussion was given by Döring [4], where an abundant list of references can also be found. The other

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convergence proofs which use the majorant principle ([7], [8]) due to Kantorovich were given by Altman [2], Safiev [11] and Grebenjuk [5].

In this paper, as an application of the technique employed by the author in a series of papers [13] - [16], we shall establish some results on convergence of the method of tangent hyperbolas under the weaker assumptions than theirs. First, in §2, we shall derive several basic results on convergence and error estimates for the method. Next, in §3, we shall apply the results to establish a semi-local convergence theorem for the method, which corresponds to the theorems for Newton's and Newton-like methods obtained in [14] and [15]. Finally, in §4, our results will be compared with those obtained by Döring [4].

2. Preliminaries

According to Safiev [11], [12], but slightly changing his notation, we assume the following, throughout this paper:

I. The operator $\Gamma = F'(x_0)^{-1}$ exists.

II. $\zeta = \|\Gamma F(x_0)\| > 0$, $M = \|\Gamma F''(x_0)\| > 0$.

III. $\|\Gamma(F''(x) - F''(y))\| \leq N\|x-y\|$, $x, y \in D_0$, $N > 0$.

IV. The equation

$$f(t) \equiv \frac{1}{6} Nt^3 + \frac{1}{2} Mt^2 - t + \zeta = 0$$

has (one negative root and) two positive roots t^*, t^{**} such that $t^* \leq t^{**}$. Equivalently

$$\zeta \leq \frac{M^2 + 4N - M\sqrt{M^2 + 2N}}{3N(M + \sqrt{M^2 + 2N})}, \tag{2.1}$$

where the equality holds if and only if $t^* = t^{**}$. (This follows from $f(\hat{t}) \leq 0$ with $\hat{t} = 2/\left(M + \sqrt{M^2 + 2N}\right)$, the positive root of $f'(t) = 0$.)

Let

$$a(t) = f'(t) - \frac{1}{2} f''(t)f'(t)^{-1}f(t)$$

and define the scalar sequence $\{t_n\}$ by

$$t_0 = 0, \; t_{n+1} = t_n - a(t_n)^{-1}f(t_n), \; n \geqq 0 \quad . \tag{2.2}$$

Then, under an additional assumption $\frac{4}{3} t^* < \hat{t}$, or, equivalently, $f(\frac{3}{4}\hat{t}) < 0$, Altman [2] proved that the sequence $\{t_n\}$ and $\{x_n\}$ are well-defined, converge to t^* and a solution x^* of (1.1) respectively, and

$$\|x_{n+1} - x_n\| \leqq t_{n+1} - t_n, \; \|x^* - x_n\| \leqq t^* - t_n, \; n \geqq 0 \quad . \tag{2.3}$$

That is, $\{t_n\}$ is a majorizing sequence for $\{x_n\}$. Grebenjuk [5] also proved (2.3) by assuming $f(2\zeta) < 0$. Furthermore, Safiev [11] proved the same result under the assumption that

$$h = M\zeta < (2 + \gamma)^{-1} \tag{2.4}$$

with $\gamma = NM^{-2}$, and

$$f''(t)f'(t)^{-2}f(t) \leqq \sigma < 2 \tag{2.5}$$

for $t \in [0,t^*]$ with a positive constant σ. As is easily seen, the condition (2.4) is stronger than (2.1), and (2.5) follows from (2.1) with $\sigma = \frac{1}{2}$, since we can show that $f'(t)^2 - 2f''(t)f(t) > 0$ for $0 \leqq t < t^*$.

In this section, we improve (2.3) under the assumption (I) - (IV). We first prepare some elementary lemmas.

Lemma 2.1. The sequence $\{t_n\}$ is well-defined and

$$0 = t_0 < t_1 < t_2 < \cdots \to t^* \quad .$$

Proof. It is easy to see that

$$f'(t) < a(t) < b \equiv - \frac{f(t)}{t^* - t} < 0$$

for $0 \leqq t < t^*$. This implies that, if t_n is defined for some $n \geqq 0$ and $0 \leqq t_n < t^*$, then

$$t_n < \tilde{t}_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)} < t_{n+1} < t^* \quad .$$

(Consider three lines which pass through the point $(t_n, f(t_n))$ with the slopes $f'(t_n)$, $a(t_n)$ and b. Then the lines intersect with the t-axis at $(\tilde{t}_{n+1}, 0)$, $(t_{n+1}, 0)$ and $(t^*, 0)$.) Starting with $t_0 = 0$, we can repeat the above argument. Q.E.D.

Remark 2.1. By Brown's remark [3], the sequence $\{t_n\}$ is identical to Newton's sequence applied to the function $g(t) = -f(t)/\sqrt{-f'(t)}$. This gives us another proof for Lemma 2.1, since we can prove that $g(t) < 0$, $g'(t) > 0$ and $g''(t) < 0$ for $0 \leqq t < t^*$. Also see Alefeld [1].

Lemma 2.2. The iterates (1.2) are well-defined for every $n \geqq 0$ and converge to a solution x^* of (1.1). More precisely, we have (2.3).

Proof. The same proof as in Altman [2] and Safiev [11] works, since, under our assumptions (I) - (IV), $\Gamma_n = F'(x_n)^{-1}$ exists for every $n \geqq 0$ and we have

$$\|\Gamma\, F''(x_n)\| \leqq f''(t_n) \quad , \tag{2.6}$$

$$\|\Gamma_n F'(x_0)\| \leqq -f'(t_n)^{-1} \quad , \tag{2.7}$$

$$\|\Gamma\, F(x_n)\| \leqq f(t_n) \quad . \tag{2.8}$$

Q.E.D.

Lemma 2.3. The following inequalities hold:

$$\|\Gamma\, F''(x_n + td_n)\| \leqq f''(t_n + t\Delta t_n) \quad ,$$

$$\|\Gamma\, F''(x_n + t(x^* - x_n))\| \leqq f''(t_n + t(t^* - t_n)) \quad ,$$

$$\|\Gamma\{F''(x_n + td_n) - F''(x_n)\}\| \leqq f''(t_n + t\Delta t_n) - f''(t_n)$$

and

$$\|\Gamma\{F''(x_n + t(x^* - x_n)) - F''(x_n)\}\| \leqq f''(t_n + t(t^* - t_n)) - f''(t_n) \quad ,$$

where $d_n = x_{n+1} - x_n$, $\Delta t_n = t_{n+1} - t_n$ and $0 \leqq t \leqq 1$.

Proof. We have

$$\|\Gamma F''(x_n + td_n)\| \leq \|\Gamma\{F''(x_n + td_n) - F''(x_0)\}\| + \|\Gamma F''(x_0)\|$$

$$\leq N\{t\|x_{n+1} - x_0\| + (1-t)\|x_n - x_0\|\} + M$$

$$\leq N\{t\, t_{n+1} + (1-t)t_n\} + M$$

$$= f''(t_n + t\Delta t_n) \quad , \quad \text{etc.}$$

Q.E.D.

Lemma 2.4. We have

$$\frac{\|x^* - x_{n+1}\|}{t^* - t_{n+1}} \leq \left(\frac{\|x^* - x_n\|}{t^* - t_n}\right)^3 \quad , \quad n \geq 0 \quad .$$

Proof. Let $\Gamma_n = F'(x_n)^{-1}$. Then we have

$$x^* - x_{n+1} = x^* - x_n + A(x_n)^{-1}F(x_n)$$

$$= -A(x_n)^{-1}\{F(x^*) - F(x_n) - F'(x_n)(x^* - x_n) - \frac{1}{2} F''(x_n)(x^* - x_n)^2$$

$$+ (F'(x_n) - A(x_n))(x^* - x_n) + \frac{1}{2} F''(x_n)(x^* - x_n)^2\}$$

$$= -A(x_n)^{-1}F(x_0)[\int_0^1 (1-t)\Gamma\{F''(x_n + t(x^* - x_n)) - F''(x_n)\}(x^* - x_n)^2 dt$$

$$+ \Gamma\{F'(x_n) - A(x_n) + \frac{1}{2} F''(x_n)(x^* - x_n)\}(x^* - x_n)]$$

and

$$F'(x_n) - A(x_n) + \frac{1}{2} F''(x_n)(x^* - x_n)$$

$$= \frac{1}{2} F''(x_n)\Gamma_n F(x_n) + \frac{1}{2} F''(x_n)(x^* - x_n)$$

$$= \frac{1}{2} F''(x_n)\Gamma_n\{F(x_n) + F'(x_n)(x^* - x_n)\}$$

$$= -\frac{1}{2} F''(x_n)\Gamma_n F'(x_0) \int_0^1 (1-t)\Gamma F''(x_n + t(x^* - x_n))(x^* - x_n)^2 dt \quad .$$

Hence we obtain from Lemma 2.2, (2.6) - (2.8) and Lemma 2.3

$$\|x^* - x_{n+1}\| \leq -a(t_n)^{-1}[\frac{N}{6}(t^* - t_n)^3$$

$$- \frac{1}{2} f''(t_n)f'(t_n)^{-1} \int_0^1 (1-t)f''(t_n + t(t^* - t_n))(t^* - t_n)^3 dt]$$

$$\left(\frac{\|x^* - x_n\|}{t^* - t_n}\right)^3$$

$$= -a(t_n)^{-1}[\{f(t^*) - f(t_n) - f'(t_n)(t^* - t_n) - \frac{1}{2} f''(t_n)(t^* - t_n)^2\}$$

$$+ \{(f'(t_n) - a(t_n))(t^* - t_n) + \frac{1}{2} f''(t_n)(t^* - t_n)^2\}] \left(\frac{\|x^* - x_n\|}{t^* - t_n}\right)^3$$

$$= \{t^* - t_n + a(t_n)^{-1}f(t_n)\} \left(\frac{\|x^* - x_n\|}{t^* - t_n}\right)^3$$

$$= (t^* - t_{n+1}) \cdot \left(\frac{\|x^* - x_n\|}{t^* - t_n}\right)^3 .$$

Q.E.D.

Lemma 2.5. We have

$$t^* - t_{n+1} < \frac{1}{3}(t^* - t_n) \quad , \quad n \geq 0 \quad , \tag{2.9}$$

or

$$t^* - t_{n+1} < \frac{1}{2}\Delta t_n \quad , \quad n \geq 0 \quad . \tag{2.10}$$

Proof. The inequality (2.9) is equivalent to

$$\frac{2}{3}(t^* - t_n) + \frac{f(t_n)}{a(t_n)} < 0 \quad .$$

If $t^* = t^{**}$, then, considering Taylor's expansion of $f(t)$, $f'(t)$ and $f''(t)$ about t^*, we can easily prove that

$$\frac{2}{3}(t^* - t) + \frac{f(t)}{a(t)} < 0 \quad , \quad 0 \leq t < t^* \quad .$$

Therefore, (2.9) holds in this case. If $t^* < t^{**}$, then there exist constants $\tilde{M}$ and $\tilde{N}$ such that $\tilde{M} > M$, $\tilde{N} > N$ and the equation $\tilde{f}(t) \equiv \frac{1}{6}\tilde{N}t^3 + \frac{1}{2}\tilde{M}t^2 - t + \zeta = 0$ has positive double roots $\tilde{t}^* = \tilde{t}^{**}$. Let $\{\tilde{t}_n\}$

be the sequence obtained by applying the method of tangent hyperbolas to $\tilde{f}(t) = 0$ with $\tilde{t}_0 = 0$. Then, $f(t) \leqq \tilde{f}(t)$ for $t \geqq 0$ and an application of the majorant theory (Lemma 2.2) implies that $t_{n+1} - t_n \leqq \tilde{t}_{n+1} - \tilde{t}_n$ and $t^* - t_n \leqq \tilde{t}^* - \tilde{t}_n$. Furthermore, we have from Lemma 2.4

$$\frac{t^* - t_{n+1}}{\tilde{t}^* - \tilde{t}_{n+1}} \leqq \left(\frac{t^* - t_n}{\tilde{t}^* - \tilde{t}_n}\right)^3 \leqq \frac{t^* - t_n}{\tilde{t}^* - \tilde{t}_n} \quad , \ n \geqq 0 \quad .$$

This leads to

$$\frac{t^* - t_{n+1}}{t^* - t_n} \leqq \frac{\tilde{t}^* - \tilde{t}_{n+1}}{\tilde{t}^* - \tilde{t}_n} < \frac{1}{3} \quad .$$

Q.E.D.

We are now in a position to prove the following:

Theorem 2.1. Let $\tau_n{}^*$ and $\sigma_n{}^*$ be the smallest positive root and the unique positive root of the equations

$$\varphi_n(t) \equiv \kappa_n t^3 - t + \delta_n = 0$$

and

$$\psi_n(t) \equiv \kappa_n t^3 + t - \delta_n = 0 \quad ,$$

respectively, where $\kappa_n = (t^* - t_{n+1})/(t^* - t_n)^3$ and $\delta_n = \|x_{n+1} - x_n\| > 0$. Then we have

$$\sigma_n{}^* \leqq \|x^* - x_n\| \leqq \tau_n{}^* \quad ,$$

and

$$\|x^* - x_{n+1}\| \leqq \tau_n{}^* - \delta_n \quad , \quad n \geqq 0 \quad .$$

Proof. By Lemma 2.4 we have

$$\|x^* - x_n\| - \delta_n \leqq \|x^* - x_{n+1}\| \leqq \kappa_n \|x^* - x_n\|^3 \quad ,$$

which implies $\varphi_n(\|x^* - x_n\|) \geqq 0$. The function $\varphi_n(t)$ attains the local minimum at $t = \tau_n \equiv \sqrt{(3\kappa_n)^{-1}}$ and, by Lemma 2.5, we have $t^* - t_n < \tau_n$. Furthermore,

$$\varphi_n(t^* - t_n) = \delta_n - \Delta t_n \leqq 0 \quad .$$

Hence the equation $\varphi_n(t) = 0$ has distinct positive roots τ_n^*, τ_n^{**} such that $\tau_n^* < \tau_n^{**}$, and we should have $\varphi_n(\tau_n) < 0$, which implies

$$\|x^* - x_n\| \leqq \tau_n^* \leqq t^* - t_n < \tau_n < \tau_n^{**} \quad ,$$

since it is known by Lemma 2.2 that $\|x^* - x_n\| \leqq t^* - t_n$. To obtain lower bounds, we use Gragg-Tapia's technique: The inequalities $\delta_n - \|x^* - x_n\| \leqq \|x^* - x_{n+1}\| \leqq \kappa_n \|x^* - x_n\|^3$ mean $\psi_n(\|x^* - x_n\|) \geqq 0$, from which we obtain $\|x^* - x_n\| \geqq \sigma_n^*$, where σ_n^* denotes the unique positive root of the equation $\psi_n(t) = 0$. Finally we have

$$\|x^* - x_{n+1}\| \leqq \kappa_n \|x^* - x_n\|^3 \leqq \kappa_n \tau_n^{*3} = \tau_n^* - \delta_n, \; n \geqq 0 \quad .$$

Q.E.D.

Corollary 2.1.1. The follwoing error estimates hold:

$$0.89\,\delta_n < \|x^* - x_n\| < 1.5\,\delta_n \quad , \quad n \geqq 0 \quad , \tag{2.11}$$

and

$$\|x^* - x_{n+1}\| < 0.5\,\delta_n \quad . \tag{2.12}$$

Proof. Let τ_n be as defined in the proof of Theorem 2.1. Then $\varphi_n(\tau_n) = \delta_n - \frac{2}{3}\tau_n < 0$ mean $\tau_n > \frac{3}{2}\delta_n$. Hence we have $\kappa_n \delta_n^2 < \frac{4}{2}$ and

$$\varphi(\tfrac{3}{2}\delta_n) < \frac{4}{27\delta_n^2}\left(\tfrac{3}{2}\delta_n\right)^3 - \tfrac{3}{2}\delta_n + \delta_n = 0$$

so that $\tau_n^* < \frac{3}{2}\delta_n$ and $\|x^* - x_{n+1}\| \leqq \tau_n^* - \delta_n < \frac{1}{2}\delta_n$. Next, let $\bar{\sigma}_n^*$ be the positive root of the equation

$$\bar{\psi}_n(t) \equiv \frac{4}{27\delta_n^2} t^3 + t - \delta_n = 0 \quad .$$

Then we have $\bar{\sigma}_n^* < \sigma_n^*$ since $\bar{\psi}_n(t) > \psi_n(t)$ for $t > 0$. The proof is completed by verifying $\bar{\psi}(0.89\,\delta_n) < 0$.

3. Main Theorem

On the basis of the results obtained in the previous section, we can now prove the following Newton-Kantorovich type theorem:

Theorem 3.1. In addition to the assumption (I) - (IV), assume that $\bar{S} = \bar{S}(x_1, t^* - t_1) = \{x \in X \mid \|x - x_1\| \leqq t^* - t_1\} \subseteq D_0$. Then:

(i) The iterates (1.2) are well-defined for every $n \geqq 0$, lie in S (interior of $\bar{S}$) for $n \geqq 1$ and converge to a solution x^* of the equation (1.1).

(ii) The solution is unique in

$$\tilde{S} = \begin{cases} S(x_0, t^{**}) \cap D_0 & (\text{if } t^* < t^{**}) \\ \bar{S}(x_0, t^{**}) \cap D_0 & (\text{if } t^* = t^{**}) \end{cases} .$$

(iii) Error estimates

$$\sigma_n^* \leqq \|x^* - x_n\| \leqq \tau_n^*$$

$$\leqq \delta_n + \frac{t^* - t_{n+1}}{(\Delta t_n)^3} \delta_n^3$$

$$\leqq \frac{t^* - t_{n+1}}{\Delta t_n} \delta_n \leqq t^* - t_n \quad , \quad n \geqq 0$$

and

$$\|x^* - x_n\| \leqq \frac{t^* - t_n}{(\Delta t_{n-1})^3} \delta_{n-1}^3 \quad , \quad n \geqq 1$$

hold, where σ_n^* and τ_n^* are defined in Theorem 2.1.

Proof. (i) was already proved in Lemmas 2.1 and 2.2. To prove (ii), let $\tilde{x}^*$ be a solution in $\tilde{S}$. Then, replacing x^* and t^* in the proof of Lemma 2.4 by $\tilde{x}^*$ and t^{**}, respectively, we have

$$\frac{\|\tilde{x}^* - x_{n+1}\|}{t^{**} - t_{n+1}} \leqq \left(\frac{\|\tilde{x}^* - x_n\|}{t^{**} - t_n}\right)^3 \leqq \cdots \leqq \left(\frac{\|\tilde{x}^* - x_0\|}{t^{**}}\right)^{3^{n+1}} \equiv \rho^{3^{n+1}} .$$

(Observe that

$$
\begin{aligned}
\|\Gamma\, F''(x_n+t(\tilde{x}^*-x_n))\| &\leqq \|\Gamma\{F''(x_n+t(\tilde{x}^*-x_n)) - F''(x_0)\}\| + \|\Gamma\, F''(x_0)\| \\
&\leqq N\{t\|\tilde{x}^*-x_0\| + (1-t)\|x_n-x_0\|\} + M \\
&\leqq N\{tt^{**} + (1-t)t_n\} + M \\
&= f''(t_n+t(t^{**} - t_n)), \quad \text{etc.} \quad)
\end{aligned}
$$

If $t^* < t^{**}$, then $\rho < 1$. If $t^* = t^*$, then $\rho \leqq 1$ and $t^{**} - t_{n+1} = t^* - t_{n+1} \to 0$ as $n \to \infty$. Therefore, in any case, we have

$$\|\tilde{x}^* - x_{n+1}\| \leqq (t^{**} - t_{n+1})\rho^{3^{n+1}} \to 0$$

as $n \to \infty$. This implies $\tilde{x}^* = \lim_{n\to\infty} x_n = x^*$.

Finally, to prove (iii), we see that

$$
\begin{aligned}
&\varphi_n\Big(\delta_n + (t^* - t_{n+1})\Big(\frac{\delta_n}{\Delta t_n}\Big)^3\Big) \\
&\leqq \frac{t^* - t_{n+1}}{(t^* - t_n)^3}\Big(\delta_n + \frac{t^* - t_{n+1}}{\Delta t_n}\,\delta_n\Big)^3 - \frac{t^* - t_{n+1}}{(\Delta t_n)^3}\,\delta_n^{\,3} = 0 .
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
\tau_n^* &\leqq \delta_n + (t^* - t_{n+1})\Big(\frac{\delta_n}{\Delta t_n}\Big)^3 \\
&\leqq \delta_n + \frac{t^* - t_{n+1}}{\Delta t_n}\,\delta_n = \frac{t^* - t_n}{\Delta t_n}\,\delta_n \leqq t^* - t_n
\end{aligned}
$$

and

$$\|x^* - x_{n+1}\| \leqq \tau_n^* - \delta_n \leqq (t^* - t_{n+1})\Big(\frac{\delta_n}{\Delta t_n}\Big)^3 , \quad n \geqq 0 .$$

Q.E.D.

Corollary 3.1.1. The following error estimates hold:

$$\|x^* - x_n\| < \left\{1 + \frac{1}{2}\left(\frac{\delta_n}{\Delta t_n}\right)^2\right\}\delta_n \leq \frac{3}{2}\delta_n \quad , \tag{3.1}$$

$$\|x^* - x_{n+1}\| < \frac{1}{2}\left(\frac{\delta_n}{\Delta t_n}\right)^2 \delta_n \leq \frac{1}{2}\delta_n \quad , \quad n \geq 0 \quad , \tag{3.2}$$

provided that $\delta_n > 0$.

Remark 3.1. Choose constants $\overline{N}$, $\overline{M}$ and $\overline{\zeta}$ such that $\overline{N} \geq N$, $\overline{M} \geq M$, $\overline{\zeta} \geq \zeta$ and the equation $\overline{f}(t) \equiv \frac{1}{6}\overline{N}t^3 + \frac{1}{2}\overline{M}t^2 - t + \overline{\zeta} = 0$ has positive solutions. Define the sequence $\{\overline{t}_n\}$ as the sequence generated by the method of tangent hyperbolas applied to the equation $\overline{f}(t) = 0$ with $\overline{t}_0 = 0$. Then, by the majorant theory, we have $\Delta t_n \leq \Delta\overline{t}_n$. Hence, if we replace Δt_n in (3.1) and (3.2) by $\Delta\overline{t}_n$, then sharper error bounds will be obtained.

4. Comparisons

It would be interesting to compare our results with known ones. We first observe that the conditions (I)- (IV) imply that

$$\begin{aligned} \|\Gamma\, F''(x)\| &\leq \|\Gamma(F''(x) - F''(x_0))\| + \|\Gamma\, F''(x_0)\| \\ &\leq N\|x - x_0\| + M \qquad\qquad (4.1) \\ &\leq Nr + M \end{aligned}$$

for every $x \in \overline{S}(x_0,r) \subseteq D_0$, where r is a positive constant.

As a modification of Mertvecova [9] and Safiev [12], Döring essentially proved [4; Satz 2.1] that if $\overline{S} = \overline{S}(x_0,2\delta_0) \subseteq D_0$, $\|\Gamma\, F''(x)\| \leq K$ for every $x \in \overline{S}$, $K\delta_0 \leq \frac{1}{2}$ and $N\delta_0{}^2 \leq \frac{21}{50}$, then $\{x_n\}$ converges to a solution x^* and

$$\|x^* - x_n\| \leq 2\delta_n$$

$$\leq \frac{2\beta_n\delta_{n-1}(\frac{1}{6} N\delta_{n-1}^2 + \frac{1}{2} K\|d_{n-1} - c_{n-1}\|)}{1 - \frac{2}{5}\eta_{n-1}}, \quad n \geq 1 \quad , \tag{4.2}$$

where $\{\beta_n\}$ and $\{\eta_n\}$ are defined by

$$\beta_0 = 1, \ \eta_0 = K\delta_0, \ \beta_n = \frac{\beta_{n-1}}{1-\eta_{n-1}}, \ \eta_n = K\beta_n\delta_n, \ n \geq 1 \quad .$$

Furthermore, if $\|\Gamma F'(x_1)^{-1}\| \leq 1$, then the solution is unique in $\overline{S}$. Under our assumptions (I) - (IV), we can choose $K = 2N\delta_0 + M$ (cf. (4.1)). Then, the condition $K\delta_0 \leq \frac{1}{2}$ is equivalent to $4N\delta_0^2 + 2M\delta_0 \leq 1$, and

$$f(2\delta_0) = \frac{4}{3} N\delta_0^2 + M\delta_0 - \delta_0 < (\frac{1}{3} + \frac{1}{2} - 1)\delta_0 = -\frac{1}{6}\delta_0 < 0 \quad .$$

Hence we obtain $t^* < 2\delta_0 < t^{**}$. It now follows from Theorem 3.1 and Corollary 2.1.1 that the solution is unique in $\tilde{S} = S(x_0, t^{**}) \cap D_0$, hence, in $\overline{S} = \overline{S}(x_0, 2\delta_0)$ and $\|x^* - x_n\| < \frac{3}{2}\delta_n$. Therefore we can replace the factor $2\beta_n$ in Döring's bound (4.2) by the smaller factor $\frac{3}{2}\beta_n$.

Döring also proved the following [4; Satz 3.1]: If $\overline{S} = \overline{S}(x_0, \frac{5}{8}\zeta) \subseteq D_0$, $\|\Gamma F''(x)\| \leq K$ for every $x \in \overline{S}$, $3K\zeta \leq 1$ and $3N\zeta^2 \leq 1$, then $\{x_n\}$ converges to a solution, which in unique in $\overline{S}$ and

$$\|x^* - x_n\| \leq \frac{8}{5}\zeta_n \tag{4.3}$$

$$\leq \frac{8}{5}\beta_n(\frac{1}{6} N\delta_{n-1}^3 + \frac{1}{2} K\|d_{n-1} - c_{n-1}\|\delta_{n-1}), \quad n \geq 1 \quad , \tag{4.4}$$

$$\leq \frac{8}{5}\beta_n(\frac{1}{6} N + \frac{1}{4}\beta_n K^2)\delta_{n-1}^3, \quad n \geq 1 \quad , \tag{4.5}$$

where $c_n = -\Gamma_n F'(x_n)$, $d_n = x_{n+1} - x_n$, $\zeta_n = \|c_n\|$ and $\{\beta_n\}$ is defined by

$$\beta_0 = 1, \ \eta_0 = \frac{6}{5} K\zeta, \ \beta_n = \frac{\beta_{n-1}}{1-\eta_{n-1}}, \ \eta_n = \frac{6}{5}\beta_n K\zeta_n, \ n \geq 1 \quad .$$

Choose $K = \frac{8}{5} N\zeta + M$ according to (4.1). Then, the condition $3K\zeta \leq 1$ is equivalent to $24N\zeta^2 + 15M\zeta \leq 5$ (the condition $3N\zeta^2 \leq 1$ is then automatically satisfied), from which we obtain

$$\begin{aligned} f(\tfrac{8}{5}\zeta) &= \frac{1}{6} N(\frac{8}{5}\zeta)^3 + \frac{1}{2} M(\frac{8}{5}\zeta)^2 - \frac{3}{5}\zeta \\ &\leq \{\frac{1}{6}(\frac{8}{5})^3 \cdot \frac{5}{24} + \frac{1}{2}(\frac{8}{5})^2 \cdot \frac{1}{3} - \frac{3}{5}\}\zeta \\ &= -\frac{18}{150}\zeta < 0 \end{aligned}$$

so that $t^* < \frac{8}{5}\zeta < t^{**}$. Hence we have from Theorem 3.1 that the solution is unique in $\overline{S} \subset \tilde{S} = S(x_0, t^{**}) \cap D_0$. Under his conditions, Döring obtained $\delta_n \leq \frac{6}{5}\zeta_n$, while we have from Corollary 3.1.1

$$\|x^* - x_n\| < \left\{1 + \frac{1}{2}\left(\frac{\delta_n}{\Delta t_n}\right)^2\right\}\delta_n$$

provided that $\delta_n \neq 0$. This, together with his estimates, leads to another estimate

$$\|x^* - x_n\| < \frac{6}{5}\left\{1 + \frac{1}{2}\left(\frac{\delta_n}{\Delta t_n}\right)^2\right\}\zeta_n \quad (n \geq 0) \tag{4.6}$$

$$\leq \frac{6}{5}\beta_n\left\{1 + \frac{1}{2}\left(\frac{\delta_{n-1}}{\Delta t_{n-1}}\right)^4\right\}\left(\frac{1}{6}N\delta_{n-1}^2 + \frac{1}{2}K\|d_{n-1} - c_{n-1}\|\right)\delta_{n-1} \quad (n \geq 1) \tag{4.7}$$

$$\leq \frac{6}{5}\beta_n\left\{1 + \frac{1}{2}\left(\frac{\delta_0}{t_1}\right)^{2^{n+1}}\right\}\left(\frac{1}{6}N + \frac{1}{4}\beta_n K^2\right)\delta_{n-1}^3 \quad (n \geq 1), \tag{4.8}$$

where we have used the inequalities

$$\frac{\delta_n}{\Delta t_n} \leq \left(\frac{\delta_{n-1}}{\Delta t_{n-1}}\right)^2 \leq \cdots \leq \left(\frac{\delta_0}{t_1}\right)^{2^n},$$

which can be proved as follows:

$$
\begin{aligned}
d_n = x_{n+1} - x_n \\
&= -A(x_n)^{-1}\{F(x_n) - F(x_{n-1}) - F'(x_{n-1})d_{n-1} - \tfrac{1}{2}F''(x_{n-1})d_{n-1}^2 \\
&\quad + (F'(x_{n-1}) - A(x_{n-1}))d_{n-1} + \tfrac{1}{2}F''(x_{n-1})d_{n-1}^2\} \\
&= -A(x_n)^{-1}F'(x_0)[\textstyle\int_0^1(1-t)\Gamma\{F''(x_{n-1}+td_{n-1}) - F''(x_{n-1})\}d_{n-1}^2\,dt \\
&\quad + \tfrac{1}{2}\Gamma\ F''(x_{n-1})\{\Gamma_{n-1}F(x_{n-1}) + d_{n-1}\}d_{n-1}] \\
&= -A(x_n)^{-1}F'(x_0)[\textstyle\int_0^1(1-t)\Gamma\{F''(x_{n-1}+td_{n-1}) - F''(x_{n-1})\}d_{n-1}^2\,dt \\
&\quad + \tfrac{1}{4}\Gamma\ F''(x_{n-1})\Gamma_{n-1}F''(x_{n-1})\Gamma_{n-1}F(x_{n-1})d_{n-1}^2] \quad .
\end{aligned}
$$

Hence

$$
\begin{aligned}
\delta_n &\leq -a(t_n)^{-1}[\textstyle\int_0^1(1-t)\{f''(t_{n-1}+t\Delta t_{n-1}) - f''(t_{n-1})\}(\Delta t_{n-1})^2dt \\
&\quad + \tfrac{1}{4}f''(t_{n-1})^2f'(t_{n-1})^{-2}f(t_{n-1})(\Delta t_{n-1})^2]\left(\frac{\delta_{n-1}}{\Delta t_{n-1}}\right)^2 \\
&= -a(t_n)^{-1}[\{f(t_n) - f(t_{n-1}) - f'(t_{n-1})\Delta t_{n-1} - \tfrac{1}{2}f''(t_{n-1})(\Delta t_{n-1})^2\} \\
&\quad + (f'(t_{n-1}) - a(t_{n-1}))\Delta t_{n-1} + \tfrac{1}{2}f''(t_{n-1})\Delta t_{n-1}]\left(\frac{\delta_{n-1}}{\Delta t_{n-1}}\right)^2 \\
&= -a(t_n)^{-1}f(t_n)\left(\frac{\delta_{n-1}}{\Delta t_{n-1}}\right)^2 \\
&= (t_{n+1} - t_n)\left(\frac{\delta_{n-1}}{\Delta t_{n-1}}\right)^2 \quad , \quad n \geq 1 \quad .
\end{aligned}
$$

Consequently, if $\left(\frac{\delta_{n-j}}{\Delta t_{n-j}}\right)^{2^{j+1}} \leq \frac{2}{3}$ for $j = 0$ or $j = 1$, or $\left(\frac{\delta_0}{t_1}\right)^{2^{n+1}} \leq \frac{2}{3}$, then (4.6) - (4.8) improve (4.3) - (4.5).

For example, consider the simple example $F(x) = x^3 - 10$, $X = \mathbb{R}$, $x_0 = 2$, which was given by Döring [4]. Then, we have

$$\zeta = |F'(x_0)^{-1}F(x_0)| = \frac{1}{6}\ , \ M = |F'(x_0)^{-1}F''(x_0)| = 1,\ N = \frac{1}{2}\ ,$$

$$x_1 = x_0 + \frac{2}{13}\ ,\ t_1 = \frac{1}{1 - \frac{1}{2}M\zeta} = \frac{2}{11}$$

so that

$$\left(\frac{\delta_0}{t_1}\right)^4 = \left(\frac{11}{13}\right)^4 < \frac{2}{3}\ .$$

Therefore, (4.7) and (4.8) are sharper than (4.4) and (4.5) for every $n \geq 1$. Furthermore, according to Remark 3.1, we can improve the bounds (4.6) - (4.8) by choosing

$$\overline{N} = N,\ \overline{\zeta} = \zeta,\ \overline{M} = \frac{1}{\zeta}\left(\frac{1}{3} - \frac{8}{5}N\zeta^2\right) = \frac{28}{15}\ (\geq M)\ .$$

For such a choice, we have

$$\overline{t}_1 = \frac{\zeta}{1 - \frac{1}{2}\overline{M}\zeta} = \frac{15}{76}\ ,\quad \frac{\delta_0}{\overline{t}_1} = \frac{152}{195}$$

so that we obtain from (4.8)

$$\|x^* - x_2\| < \frac{6}{5}\beta_2\left\{1 + \frac{1}{2}\left(\frac{\delta_0}{\overline{t}_1}\right)^8\right\}\left(\frac{1}{6}N + \frac{1}{4}\beta_2K^2\right)\delta_1^3 < 7.994 \times 10^{-11}\ ,$$

while (4.5) gives us

$$\|x^* - x_2\| \leq 9.98 \times 10^{-11}$$

(cf. Döring [4; Table 2]).

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As an application of the technique employed by the author in a series of papers [13] - [16], some results are established concerning convergence of the method of tangent hyperbolas for solving nonlinear equations in Banach spaces as well as existence and uniqueness of solution. The results are compared with those obtained by Döring [4].

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